When Do We Know That We Need To Use Parts?

When the integral is a **product** of two completely **unrelated functions** (and we can't use reverse chain rule or substitution). Examples of this include:

- $\int$ **Power** $\times$ **Trig**   e.g. $\int x \sin x \, dx$ or $\int x^2 \sin x \, dx$
- $\int$ **Power** $\times$ **Exponential**   e.g. $\int x \, e^x dx$ or $\int x^2 e^x dx$
- $\int$ **Easy power** $\times$ **Harder power** e.g. $\int 5x\sqrt{2-x} \, dx$
- $\int$ **Power** $\times$ **Ln**   e.g. $\int x \ln x \, dx$ or $\int x^2 \ln x \, dx$ or $\int \frac{1}{x} \ln x \, dx$
- $\int$ **Exponential** $\times$ **Trig**   e.g. $\int e^x \sin x \, dx$ or $\int e^x \cos x \, dx$

There are two harder types (natural logarithm and inverse trig) that aren't a product and it is not obvious to use parts. These types need to be remembered as special cases and turned/forced into a product in order to use parts:

- $\int \ln x = \int \mathbf{1} \times \mathbf{\ln x}$
- $\int \sin^{-1} x = \int \mathbf{1} \times \mathbf{\sin^{-1} x}$
- $\int \cos^{-1} x = \int \mathbf{1} \times \mathbf{\cos^{-1} x}$
- $\int \tan^{-1} x = \int \mathbf{1} \times \mathbf{\tan^{-1} x}$

A good rule of thumb to follow if you want to know whether to use parts would be to try a substitution first (so make sure you're good at substitution before learning parts), and then if the resulting doesn't become easier to integrate once simplified, try integration by parts. A substitution is mostly used (but certainly not always) whenever we have a function f containing a function (we'll call this g) multiplied by the derivative of g. That is, if you have the form $\int g'(x)f(g(x))dx$. For example,

$$\int xe^{x^2} dx$$
$$f(g(x)) = e^{x^2} so \, f = e^x, g = x^2$$

The derivative of x^2 is $2x$ and we have the correct x term. It doesn't matter if the constant 2 in front is different. In this case we would use substitution (or reverse chain rule if you're good at integration) instead of parts. **It is worth mentioning that parts would still work, it would just take longer!** So, **lots of integrals that can be done by substitution can be done with parts instead, but the converse is not true.**

The best strategy is to try to "play it out" in your mind i.e. whether substitution or parts will be better and try to see which one will work better. The only way to get better at these sorts of integrals is to practice large sets of each type. Then, you start to think "Oh—this looks like a substitution!" or, "maybe by-parts is better for this." **Practise, practise, practise is really the best way to get better at recognizing whether to use substitution or parts and is key to becoming good at parts.**

How Do We Apply Parts?

We say our original integral has the form/looks like $\int u \frac{dv}{dx} dx$ or $\int \frac{dv}{dx} u \, dx$
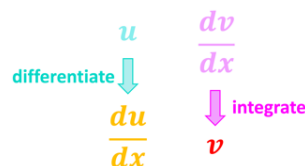
Step 1:

Call one of the functions u and the other function $\frac{dv}{dx}$. It matters which we call u and which we call $\frac{dv}{dx}$!

See the table on the bottom of the next page for how to choose which function is u and which is $\frac{dv}{dx}$

Step 2:

Find $\frac{du}{dx}$ by differentiating u and v by integrating $\frac{dv}{dx}$

$$u \qquad \frac{dv}{dx}$$
differentiate $\downarrow$ $\qquad \downarrow$ integrate
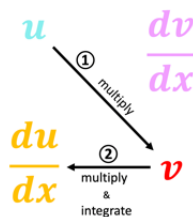$$\frac{du}{dx} \qquad v$$

Notice how for both functions we want to end up with the original function and its corresponding derivative which leads us to step 3.

Step 3:

Plug into the formula $uv - \int v \frac{du}{dx} dx$

It can help to remember the formula as $①$ $-$ $②$

$$u \qquad \frac{dv}{dx}$$
$$① \text{ multiply}$$
$$\frac{du}{dx} \qquad v$$
$$② \text{ multiply & integrate}$$

Step 4:

Simplify the integral part $\int v \frac{du}{dx} dx$ as much as possible before integrating. You take out constants (that are multiplied or divided) to make the integration easier.

Step 1 continued:

How to choose which function is u and which function is $\frac{dv}{dx}$?

We need to call u the function that we can differentiate and $\frac{dv}{dx}$ the function which we can EASILY integrate. Recall that our functions are either algebraic functions i.e. x^n, trig functions, logarithmic functions, exponentials or inverse trig functions. There are many 3 ways you can decide which function to use for u and which function to use for $\frac{dv}{dx}$.

Way 1: Best Method For Beginners	Way 2:	Way 3:
Use the acronym **LIATE**	Memorise the following templates for the different types of functions	Think ahead and play it out what will happen in mind

Way 1: Best Method For Beginners — Use the acronym LIATE

Consider all the possible types of functions that can come up:

Logarithms $\quad \ln x, \log x$

Inverse Trig $\quad \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$

Algebra $\quad$ constants, $x, x^2, ...$

Trigonometry $\quad \sin x, \cos x, \tan x$

Exponentials $\quad e^x$

Locate the two functions given inside in the integral. Whichever of the 2 functions that comes first on the list above is always called u.

For example:

$$\int x^3 \ln x\, dx$$

x^3 is the **A**lgebra function
$\ln x$ is the **L**ogarithmic function
The Logarithmic function (L) comes before Algebra (A) on the list above so we call logarithmic function u and the algebraic function $\frac{dv}{dx}$

This method doesn't always work for harder questions, such as when both functions are trig, but it is a great method to help beginners start to get a feel for integration by parts.

Way 2:

blue functions below are u
and
pink functions below are $\frac{dv}{dx}$

The types we encounter are:

- $\int (Power)(Trig)$
  $eg. \int (x^2)(\sin x)$

- $\int (Power)(Exponential)$
  $eg. \int (x^2)(e^x)$

- $\int (Easy\ power)(Harder\ power)$
  $eg. \int x\sqrt{x-4}$
  $eg. \int (x+2)(x+5)^5$

- $\int (Easy\ power)(ln)$
  $eg. \int (x^2)(\ln 2x)$

- $\int (Exp)(Trig)$ or $\int (Exp)(Trig)$
  (the order you pick the functions actually doesn't matter for this type)

Two harder types that aren't a product which need to be turned into a product:

$$\int ln = \int 1 \times ln$$
$$\int Inverse\ trig = \int 1 \times Inverse\ trig$$

Way 3:

This way comes naturally once you have more experience.

We decide which function to call "u" first. This is the function that will become an "easier" function when differentiated. For example

$$\int x^3 \sin x\, dx$$

x^3 becomes $3x^2$ when differentiated i.e. when we differentiate "u" we "kill" powers. $\sin x$ becomes $\cos x$ when differentiated. This is not simpler. We always look to make $\frac{du}{dx}$ simpler than "u", since this will make the resulting latter integral part of the formula $\int v \frac{du}{dx} dx$ easier to integrate.

The other function is then automatically $\frac{dv}{dx}$. Remember the function that you call $\frac{dv}{dx}$ is the function you must be able to integrate, so never call a function $\frac{dv}{dx}$ that you are unable integrate.

Note: Another way to think of this is to figure out what you need to integrate first, since finding the derivative is the easiest, whereas integration is a bit 'more fussy'. You should always be able to differentiate either function, but not necessarily integrate either, so make sure you choose the correct one to integrate!

Note: There is an integration shortcut hack called the DI method which avoids having to use the formula $uv - \int v \frac{du}{dx} dx$ altogether. See my integration by parts 'DI Integration By Parts Shortcut Method' cheat sheet for this.

Example

$$\int x \sin 2x \, dx$$

Here the integral is a **product** of two completely **unrelated functions**

$$\int \textbf{Power} \times \textbf{Trig}$$

Step 1:

We look at the functions inside the integral and we have to call one of the functions u and the other function $\frac{dv}{dx}$.

It matters which we call u and which we call $\frac{dv}{dx}$! To choose which function is u and which is $\frac{dv}{dx}$ we can use the following criteria **LIATE**

Logarithms $\quad \ln x, \log x$

Inverse Trig $\quad \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$

Algebra $\quad constants, x, x^2, \dots$

Trigonometry $\quad \sin x, \cos x, \tan x$

Exponentials $\quad e^x$

Locate the two functions given inside in the integral. Whichever function out of the 2 functions comes first on the list above is always called u.
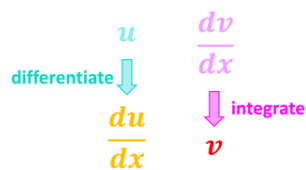
In this question we have

$$\int x \sin 2x \, dx$$

x is the **A**lgebra function
$\sin 2x$ is the **T**rig function

Algebra Function (A) comes before the Trig function (T) on the LIATE list above so we call the Algebra function u and the Trig function $\frac{dv}{dx}$.

Step 2:

Find $\frac{du}{dx}$ by differentiating u and v by integrating $\frac{dv}{dx}$

$u \qquad \frac{dv}{dx}$

differentiate $\Downarrow \qquad \qquad \Downarrow$ integrate

$\frac{du}{dx} \qquad v$

So, in our case we get

$$\int x \sin 2x \, dx$$

$$u = x \qquad \frac{dv}{dx} = \sin 2x$$

$$\frac{du}{dx} = 1 \qquad v = \frac{-\cos 2x}{2} = -\frac{1}{2}\cos 2x$$

Notice how for both functions we want to end up with the original function and its corresponding derivative which leads us to step 3.

Step 3:

Plug into the formula $uv - \int v \frac{du}{dx} dx$

Some students struggle to remember this formula. It can help to remember the formula as ① − ② in the pattern below:

$$u \qquad \frac{dv}{dx}$$

① multiply

$$\frac{du}{dx} \qquad ② \quad v$$

multiply
&
integrate
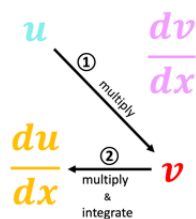
So, plugging into $uv - \int v \frac{du}{dx} dx$ gives

$$= x\left(-\frac{1}{2}\cos 2x\right) - \int \left(-\frac{1}{2}\cos 2x\right)(1)dx$$

Step 4:

Simplify the integral part $\int v \frac{du}{dx} dx$ as much as possible before integrating. You take out constants (that are multiplied or divided) to make the integration easier.

Simplifying gives

$$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$$

Taking out the constant $-\frac{1}{2}$ to make it easier:

$$= -\frac{1}{2}x\cos 2x - \left(-\frac{1}{2}\right)\int \cos 2x \, dx$$

Integrating (always remember the constant of integration + c)

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + c$$

Simplifying:

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$